THEOREM: THE CONSTANT RULE

The derivative of a constant function is zero. That is, if c is a real number, then

$$\frac{d}{dx}[c] = 0$$

Example 1: Find the derivative of the function $\frac{d}{dx} g(x) = \frac{d}{dx} -5$.

$$\boxed{g'(x) = 0}$$

THEOREM: THE POWER RULE

If n is a rational number, then the function $f(x) = x^n$ is differentiable and

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1}$$

For f to be differentiable at $x = 0$, n must be a number such that x^{n-1} is defined on an interval containing zero.

Example 2: Find the following derivatives.

a. $\frac{d}{dx} f(x) = \frac{d}{dx} x^{-5}$

$$f'(x) = \boxed{-5x^{-6} = -\frac{5}{x^6}}$$

b. $\frac{d}{dx} f(x) = \frac{d}{dx} x^{1/2}$

$$f'(x) = \boxed{\frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}}$$

THEOREM: THE CONSTANT MULTIPLE RULE

If f is a differentiable function and *c* is a real number, then cf is also differentiable and

$$\frac{d}{dx}\left[cf(x)\right] = cf'(x)$$

Example 3: Find the slope of the graph of $f(x) = 4x^{2/3}$ at

a. $x = x$

$$\frac{d}{dx} f(x) = \frac{d}{dx} 4x^{2/3}$$

$$f'(x) = 4 \cdot \frac{2}{3} x^{-1/3} = \boxed{\frac{8}{3x^{1/3}}}$$

b. $x = 125$

$$f'(x) = \frac{8}{3\sqrt[3]{x}}$$

$$f'(125) = \frac{8}{3\sqrt[3]{125}}$$

$$\boxed{f'(125) = \frac{8}{15}}$$

c. $x = -64$

$$f'(-64) = \frac{8}{3\sqrt[3]{-64}}$$

$$\boxed{f'(-64) = -\frac{2}{3}}$$

THEOREM: THE SUM AND DIFFERENCE RULES

The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f+g$ (or $f-g$) is the sum (or difference) of the derivatives of f and g.

$$\frac{d}{dx}\left[f(x)+g(x)\right]=f'(x)+g'(x)$$

$$\frac{d}{dx}\left[f(x)-g(x)\right]=f'(x)-g'(x)$$

Example 4: Find the equation of the line tangent to the graph of $f(x)=x-\sqrt{x}$ at x = 4.

1. Find $f'(x)$ at $x=x$.

$$\frac{d}{dx}f(x)=\frac{d}{dx}\left(x-x^{1/2}\right)$$

$$f'(x)=\frac{d}{dx}x-\frac{d}{dx}x^{1/2}$$

$$f'(x)=1-\frac{1}{2}x^{-1/2}$$

$$f'(x)=1-\frac{1}{2\sqrt{x}}$$

2. Find slope at $x=4$.

$$f'(4)=1-\frac{1}{2\sqrt{4}}$$

$$f'(4)=\frac{3}{4}$$

3. Find the equation of the line tangent to the graph at $x=4$ using the point-slope form of the equation of the line.

Find $f(4)$: $f(x)=x-\sqrt{x}$

$$f(4)=4-\sqrt{4}$$

$$f(4)=2$$

so $(4,2)$ is a point on f and on the tangent line

$$y-y_1=m(x-x_1)$$

$$\boxed{y-2=\frac{3}{4}(x-4)}$$

THEOREM: DERIVATIVES OF THE TRIGONOMETRIC FUNCTIONS

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cos x] = -\sin x$$
$$\frac{d}{dx}[\csc x] = -\csc x \cot x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$
$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

Example 5: Find the derivative of the following functions:

a. $\frac{d}{dx} f(x) = \frac{\sin x}{6} = \frac{d}{dx} \frac{1}{6} \sin x$

$\boxed{f'(x) = \frac{1}{6} \cos x}$

b. $\frac{d}{d\theta} r(\theta) = \frac{d}{d\theta}(5\theta - 3\cos\theta)$

$r'(\theta) = 5 - 3(-\sin\theta)$

$\boxed{r'(\theta) = 5 + 3\sin\theta}$

c. $h(t) = \frac{\cos t}{\cot t} = \frac{\cos t}{\frac{\cos t}{\sin t}} = \cancel{\cos t} \cdot \frac{\sin t}{\cancel{\cos t}} = \sin t$

$\frac{d}{dt} h(t) \frac{d}{dt} \sin t \longrightarrow \boxed{h'(t) = \cos t}$

$\boxed{\sec x = \frac{1}{\cos x} = (\cos x)^{-1}}$

d. $\frac{d}{dx} f(x) = \frac{d}{dx}(12 + 7\sec x)$

$f'(x) = 0 + 7\sec x \tan x$

$\boxed{f'(x) = 7\sec x \tan x}$

e. $\frac{d}{dx} f(x) = \frac{d}{dx}(-\tan x + x)$

$\boxed{f'(x) = -\sec^2 x + 1}$

THEOREM: THE PRODUCT RULE

The product of two differentiable functions f and g is itself differentiable. Moreover, the derivative of fg is the derivative of the first function times the second function, plus the first function times the derivative of the second function.

$$\frac{d}{dx}\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x)$$

This rule extends to cover products of more than two factors. For example the derivative of the product of functions $fghk$ is

$$\frac{d}{dx}[fghk]=f'(x)g(x)h(x)k(x)+f(x)g'(x)h(x)k(x)+f(x)g(x)h'(x)k(x)+f(x)g(x)h(x)k'(x)$$

Example 6: Find the derivative of the following functions. Simplify your result to a single rational expression with positive exponents.

a. $\frac{d}{dx}g(x)=\frac{d}{dx}x\cos x$

$g'(x)=\left(\frac{d}{dx}x\right)\cos x+x\left(\frac{d}{dx}\cos x\right)$

$g'(x)=1\cos x+x(-\sin x)$

$g'(x)=\cos x-x\sin x$

b. $\frac{d}{dt}h(t)=\left(3-\sqrt{t}\right)^2=\frac{d}{dt}\left[(3-t^{1/2})(3-t^{1/2})\right]$

$h'(t)=\left[\frac{d}{dt}(3-t^{1/2})\right](3-t^{1/2})+(3-t^{1/2})\left[\frac{d}{dt}(3-t^{1/2})\right]$

$h'(t)=\left(-\frac{1}{2}t^{-1/2}\right)(3-t^{1/2})+(3-t^{1/2})\left(-\frac{1}{2}t^{-1/2}\right)$

$h'(t)=-t^{-1/2}(3-t^{1/2})$

$\boxed{h'(t)=-\frac{(3-t^{1/2})}{t^{1/2}}}$

THEOREM: THE QUOTIENT RULE

The quotient of two differentiable functions f and g is itself differentiable at all values of x for which $g(x) \neq 0$. Moreover, the derivative of f/g is the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{\left[g(x)\right]^2}$$

Example 7: Find the derivative of the following functions. Simplify your result to a single rational expression with positive exponents.

a. $\frac{d}{dx} g(x) = \frac{d}{dx} \frac{2x}{5x^2+3}$

$$g'(x) = \frac{\left(\frac{d}{dx}2x\right)(5x^2+3) - (2x)\left[\frac{d}{dx}(5x^2+3)\right]}{(5x^2+3)^2}$$

$$g'(x) = \frac{2(5x^2+3) - 2x(10x)}{(5x^2+3)^2}$$

simplifying LTS (Left To Student)

b. $\frac{d}{dt} h(t) = \frac{d}{dt} \frac{t}{\sqrt{t}-1}$

$$h'(t) = \frac{\left(\frac{d}{dt}t\right)(t^{1/2}-1) - t\left[\frac{d}{dt}(t^{1/2}-1)\right]}{(\sqrt{t}-1)^2}$$

$$h'(t) = \frac{1(t^{1/2}-1) - t\left(\frac{1}{2}t^{-1/2}\right)}{(\sqrt{t}-1)^2}$$

$$h'(t) = \frac{t^{1/2} - 1 - \frac{1}{2}t^{1/2}}{(\sqrt{t}-1)^2}$$

$$h'(t) = \frac{\frac{1}{2}t^{1/2} - 1\,\frac{2}{2}}{(\sqrt{t}-1)^2}$$

$$\boxed{h'(t) = \frac{t^{1/2} - 2}{2(\sqrt{t}-1)^2}}$$

c. $h(t) = \frac{\cot t}{t}$

$$h'(t) = \frac{(-\csc^2 t)t - (\cot t)(1)}{t^2}$$

$$\boxed{h'(t) = \frac{-t\csc^2 t - \cot t}{t^2}}$$

Example 8: Find the given higher-order derivative.

a. $f(x)=2-\frac{2}{x}, \quad f'''(x)$

$f(x)=2-2x^{-1}$

$f'(x)=2x^{-2}$

$f''(x)=-4x^{-3}$

$\boxed{f'''(x)=12x^{-4}=\frac{12}{x^4}}$

b. $f^{(4)}(x)=2x+1, \quad f^{(6)}(x)$

$f^{(5)}(x)=2$

$\boxed{f^{(6)}(x)=0}$

Theorem: The Chain Rule

If $y=f(u)$ is a differentiable function of *u* and $u=g(x)$ is a differentiable function of *x*, then $y=f(g(x))$ is a differentiable function of *x* and

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx} \text{ or } \frac{d}{dx}\left[f(g(x))\right]=f'(g(x))g'(x).$$

Example 9: Find the derivative using the Chain Rule.

a. $y=\left(\sqrt{x}-x^2\right)^{10}$

$\frac{dy}{dx}=10\left(x^{1/2}-x^2\right)^9\left(\frac{d}{dx}\left(x^{1/2}-x^2\right)\right)$

$\frac{dy}{dx}=10\left(x^{1/2}-x^2\right)^9\left(\frac{1}{2}x^{-1/2}-2x\right)$

not simplified

b. $f(x)=\frac{1}{\sqrt{2-x}}=(2-x)^{-1/2}$

$f'(x)=-\frac{1}{2}(2-x)^{-3/2}\cdot\frac{d}{dx}(2-x)$

$f'(x)=-\frac{1}{2}(2-x)^{-3/2}(-1)$

$\boxed{f'(x)=\frac{1}{2(2-x)^{3/2}}}$

Example 10: Find the derivative of the following functions.

a. $y = \sin x$

$y' = \cos x$

b. $y = \sin 2x$

$y' = \cos(2x)\left(\frac{d}{dx} 2x\right)$

$\boxed{y' = 2\cos 2x}$

c. $y = \sin^2 x$

$y = (\sin x)^2$

$y' = 2(\sin x)^1\left(\frac{d}{dx}(\sin x)\right)$

$2\sin x \cdot \frac{1}{\cos x}$

$\to 2\frac{\sin x}{\cos x} \to 2\tan x$

$\boxed{y' = 2\sin x \cos x}$

or

$\boxed{y' = \sin 2x}$

d. $y = \sin x^2$

$y' = \cos(x^2)\left(\frac{d}{dx} x^2\right)$

$\boxed{y' = 2x\cos x^2}$

e. $y = \sqrt{\cos x} = (\cos x)^{1/2}$

$y' = \frac{1}{2}(\cos x)^{-1/2}\left[\frac{d}{dx}\cos x\right]$

$\boxed{y' = -\frac{1}{2}\sin x \cos^{-1/2} x}$

f. $f(x) = x^2(2-x)^{2/3}$

chain rule

$f'(x) = \underbrace{2x(2-x)^{2/3} + x^2\left[\frac{2}{3}(2-x)^{-1/3}\left(\frac{d}{dx}(2-x)\right)\right]}_{\text{product rule}}$

$f'(x) = 2x(2-x)^{2/3} + \frac{2}{3}x^2(2-x)^{-1/3}(-1)$

$f'(x) = \frac{2}{3}x(2-x)^{-1/3}\left[3(2-x)^1 - x\right]$

$f'(x) = \frac{2}{3}x(2-x)^{-1/3}(6-4x)$

$\boxed{f'(x) = \frac{4x(3-2x)}{3(2-x)^{1/3}}}$

g. $h(x) = x\sin^2 4x = x(\sin 4x)^2$

$h'(x) = 1(\sin 4x)^2 + x[2(\sin 4x)^1(\cos 4x)(4)]$ — $\sin 8x$

$\boxed{h'(x) = \sin^2 4x + 8x\sin 4x\cos 4x = \sin^2 4x + 4x\sin 8x}$

Theorem: Derivative of the Natural Logarithmic Function

Let *u* be a differentiable function of *x*.

1. $\dfrac{d}{dx}[\ln x] = \dfrac{1}{x}$, $x > 0$

2. $\dfrac{d}{dx}[\ln u] = \dfrac{u'}{u}$, $u > 0$

$\dfrac{d}{dx}[\ln|u|] = \dfrac{u'}{u}$, $(-\infty, 0) \cup (0, \infty)$

Example 11: Find the derivative.

a. $y = (\ln x)^3$

$y' = 3(\ln x)^2 \cdot \dfrac{1}{x}$

$\boxed{y' = \dfrac{3(\ln x)^2}{x}}$

b. $f(x) = \ln|\cos x|$

$f'(x) = \dfrac{-\sin x}{\cos x}$

$\boxed{f'(x) = -\tan x}$

c. $h(t) = \ln x^x = x\ln x$

error!

$h(x) = \ln x^x = x\ln x$

$h'(x) = 1\ln x + x \cdot \dfrac{1}{x}$

$\boxed{h'(x) = 1 + \ln x}$

Theorem: Derivative of the Natural Exponential Function

Let u be a differentiable function of x.

1. $\dfrac{d}{dx}\left[e^x\right]=e^x$

2. $\dfrac{d}{dx}\left[e^u\right]=e^u u'$

Example 12: Find the derivative.

a. $y=xe^{-x}$

$y'=1e^{-x}+xe^{-x}(-1)$

$\boxed{y'=e^{-x}(1-x)}$

b. $f(x)=e^{\sin 2x}$

$f'(x)=e^{\sin 2x}(\cos 2x)\cdot 2$

$\boxed{f'(x)=2(\cos 2x)e^{\sin 2x}}$

c. $h(t)=\dfrac{e^t}{\ln e^{\sqrt{t}}}=\dfrac{e^t}{\sqrt{t}\ln e}=\dfrac{e^t}{\sqrt{t}(1)}=\dfrac{e^t}{t^{1/2}}$

$h'(t)=\dfrac{e^t t^{1/2}-e^t\left(\frac{1}{2}t^{-1/2}\right)}{\left(t^{1/2}\right)^2}$

$h'(t)=\dfrac{\frac{1}{2}t^{-1/2}e^t\left(t^1-1\right)}{t}$

$\boxed{h'(t)=\dfrac{e^t(t-1)}{2t^{3/2}}}$

Theorem: Derivatives for Bases other than *e*

Let *a* be a positive real number $(a \neq 1)$ and let *u* be a differentiable function of *x*.

1. $\frac{d}{dx}\left[a^x\right] = (\ln a)a^x$
2. $\frac{d}{dx}\left[a^u\right] = (\ln a)a^u u'$
3. $\frac{d}{dx}\left[\log_a x\right] = \frac{1}{(\ln a)x}$
4. $\frac{d}{dx}\left[\log_a u\right] = \frac{u'}{(\ln a)u}$

Example 13: Find the derivative.

a. $y = 2^{3x}$

$y' = (\ln 2)2^{3x}(3)$

$\boxed{y' = (3\ln 2)2^{3x}}$

$y' = (\ln 2^3)(2^{3x})$

or

$\boxed{y' = (\ln 8)2^{3x}}$

b. $f(x) = \log 5x = \log_{10} 5x$

$f'(x) = \frac{\cancel{5}}{(\ln 10)(\cancel{5}x)}$

$\boxed{f'(x) = \frac{1}{x \ln 10}}$

or

$\boxed{f'(x) = \frac{1}{\ln 10}x}$

THEOREM: DERIVATIVES OF THE INVERSE TRIGONOMETRIC FUNCTIONS (*u* is a function of *x*)

$$\frac{d}{dx}[\arcsin u] = \frac{du/dx}{\sqrt{1-u^2}} \qquad \frac{d}{dx}[\arccos u] = -\frac{du/dx}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\operatorname{arccsc} u] = -\frac{du/dx}{|u|\sqrt{u^2-1}} \qquad \frac{d}{dx}[\operatorname{arcsec} u] = \frac{du/dx}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}[\arctan u] = \frac{du/dx}{1+u^2} \qquad \frac{d}{dx}[\operatorname{arccot} u] = -\frac{du/dx}{1+u^2}$$

Example 14: Find the derivative.

a. $y = \arctan 3x - \ln\left(1+9x^2\right)$

$$y' = \frac{3}{1+(3x)^2} - \frac{18x}{1+9x^2}$$

$$\boxed{y' = \frac{3-18x}{1+9x^2}}$$

b. $f(x) = x \arcsin\sqrt{x}$

$$f'(x) = 1 \arcsin\sqrt{x} + x \frac{\frac{1}{2}x^{-1/2}}{\sqrt{1-(\sqrt{x})^2}}$$

$$f'(x) = \arcsin\sqrt{x} + \frac{x}{2\sqrt{x}\sqrt{1-x}}$$

$$\boxed{f'(x) = \frac{2\sqrt{x}\sqrt{1-x}\,\arcsin\sqrt{x} + x}{2\sqrt{x}\sqrt{1-x}}}$$